Vertical Datum Unification for the International Height Reference System (IHRS)

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◆ The GBVP approach to height datum unification
  ▶ Multi-datum GBVP solution
  ▶ Observation equations and least-squares estimation
◆ Factors affecting the estimation of datum offsets – NA simulation
  ▶ Geoid omission error
  ▶ Indirect bias term
◆ Determination of datum offsets from global $W_o$ – SA example
  ▶ Available observables
  ▶ Estimation of datum parameters
◆ Conclusions
International Height Reference System (IHRS)

Definition

- Earth-fixed geopotential reference system with Coordinates:
  - geopotential values $W(x)$ (and $dW(x)/dt$)
  - geocentric Cartesian coordinates $x$ (and $dx/dt$) in the ITRS/ITRF

- Parameters, observations, data in mean-tide / mean crust system

Realization

- International Height Reference Frame (IHRF) of stations with
  - $x$, $dx/dt$
  - $W(x)$, $dW(x)/dt$
    or, preferably,
    - $W_o = \text{const.} = 62\,636\,853.4 \text{ m}^2\text{s}^{-2}$ (conventional)
    - $C(x) = -\Delta W(x) = W_o - W(x)$, $dC(x)/dt$
    - $W_p = W(x_p)$ from: levelling/altimetry + gravity; GBVP solution; hi-res GGM

- Standards, conventions, procedures
  - consistency between the definition (IHRS) and the realization (IHRF)
Illustration of the principle

Offsets between global datum and each local datum \( i \) (subscript \( i \))

\[
W_{oi} = h_P H^*_P \quad Pi \quad \text{local datum height offset}
\]

\[
W_{oi} = W_o = h_P C_{Pi} T_{Pi} \quad \text{local datum potential offset}
\]

source: Sanchez and Sideris, 2017
The GBVP Approach (1/2)

**GBVP Approach to Datum Unification**

- **The global datum:** \( P_o, \ W(P_o) = W_o, \ \Delta W_o = W_o - U_o \)
  - Problem input and output (assume \( M = M_e \)):
    
    \[
    g = \frac{\partial T}{\partial r} \left( \frac{2}{R} T + \frac{2}{R} W_o \right)
    \]
    
    \[
    T_P = W_o + T_P^{grav} = W_o + T_P^{GGM} + S_P \ g^{res}
    \]

- **The local datum:** \( P_{oi}, \ W(P_{oi}) = W_{oi}, \ \delta W_{oi} = W_o - W_{oi} \)
  - Problem input and output:
    
    \[
    g_i = g \left( \frac{2}{R} W_{oi} \right)
    \]
    
    \[
    T_{Pi} = W_o + W_{oi} + T_P^{GGM} + S_P \left( g_i^{res} + \frac{2}{R} W_{oi} \right) = W_o + W_{oi} + T_{Pi}^{grav} + T_{Pi}
    \]

- The datum potential offset, \( \delta W_{oi} \), causes both a **direct effect** on \( \zeta, \ \delta W_{oi}/\gamma \), and an **indirect effect** \( \delta T_{Pi}/\gamma \) through the biased gravity anomalies.
The GBVP Approach (2/2)

◆ Estimation of $T$ to be used for datum offset determination

The gravimetric $T$ in datum $i$ using (biased) gravity data in that datum is

$$T_{pi} = W_0 + W_{oi} + T_{p}^{GGM} + T_{pi}^{res} + T_{pi}^{ind}$$

The **indirect effect** $T_{pi}^{ind}$ is the most troublesome to compute as, for $J$ different datums, it requires knowledge of all offsets:

$$T_{pi}^{ind} = S_P^i \left( \frac{2}{R} \right) W_{oi} + \sum_{j=1 \atop j \neq i}^{J} S_P^j \left( \frac{2}{R} \right) W_{oj}$$

$$= W_{oi} S_P^i \left( \frac{2}{R} \right) + \sum_{j=1 \atop j \neq i}^{J} W_{oj} S_P^j \left( \frac{2}{R} \right) = W_{oi} f_{Pi} + \sum_{j=1 \atop j \neq i}^{J} W_{oj} f_{Pj}$$

◆ Observation equations

$$W_{oi} + T_{pi}^{ind} + \nu_P = h_P$$

$$W_{o,i} + T_{pi}^{ind} + T_{P,i+1}^{ind} + \nu_P = C_{Pi} + C_{P,i+1}$$
Least-squares Estimation of Offsets

◆ Functional model

\[ l_p = h_p C_{pi} + W_o T_{pi}^{GM} T_{pi}^{res} = (1 + f_{pi}) W_{oi} + \sum_{j=1}^{J} f_{pj} W_{oj} + \nu_p \]

\[ \ell = Ax + v \]

\[ x = (\delta W_{o1}, \delta W_{o2}, \ldots, \delta W_{oJ})^T \]

\[ \ell = (l_{p1}, l_{p2}, \ldots, l_{pN})^T , \]

\[ l_{pk} = \gamma h_{pk} - C_{pi} + \Delta W_o - T_{pk}^{GM} - T_{pi}^{res} \]

◆ Stochastic model

\[ C_l = 2 C_h + C_C + C_T \]

◆ Solution

\[ \hat{x} = (A C_l^{-1} A)^{-1} A C_l^{-1} \ell, \quad C_x = (A C_l^{-1} A)^{-1} \]
Working with residual gravity anomalies

- Remove-restore method
  \[ T_{Pi}^{\text{grav}} = T_{P}^{\text{GGM}} + T_{Pi}^{\text{res}} = T_{P}^{\text{GGM}} + S_{P}^{\text{res}} g_{i}^{\text{res}} , \quad g_{i}^{\text{res}} = g_{i} + g^{\text{GGM}} \]

- Then the residual Stokes kernel should also be used in the computation of the \( f_{Pi} \) coefficients
  \[ f_{Pi}^{\text{res}} = S_{P}^{\text{res}} \left( \frac{2}{R} \right) \]

Questions to be investigated, for cm-level datum unification:

If we use satellite only GGMs of \( N_{\text{max}} \geq 180 \),

- Will the omission error be small enough to ignore?
- Will the indirect effect be small enough to omit?
Omission error approximated by EGM2008 for $181 \leq n \leq 2190$

- Tested at NA tide gauges

<table>
<thead>
<tr>
<th>Region</th>
<th>Mean ($\text{m}^2\text{s}^{-2}$)</th>
<th>$\sigma$ ($\text{m}^2\text{s}^{-2}$)</th>
<th>Min ($\text{m}^2\text{s}^{-2}$)</th>
<th>Max ($\text{m}^2\text{s}^{-2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada Atlantic (7)</td>
<td>0.10</td>
<td>3.23</td>
<td>-5.49</td>
<td>3.72</td>
</tr>
<tr>
<td>US Atlantic (28)</td>
<td>-0.29</td>
<td>3.63</td>
<td>-8.23</td>
<td>7.06</td>
</tr>
<tr>
<td>Canada Pacific (5)</td>
<td>0.59</td>
<td>3.92</td>
<td>-3.53</td>
<td>6.27</td>
</tr>
<tr>
<td>US Pacific (17)</td>
<td>0.69</td>
<td>3.23</td>
<td>-4.80</td>
<td>7.25</td>
</tr>
<tr>
<td>Gulf of Mexico (13)</td>
<td>0.29</td>
<td>2.55</td>
<td>-6.08</td>
<td>2.74</td>
</tr>
</tbody>
</table>

- Averaging over many points reduces the omission error
- Omission error can reach several dm at individual stations and therefore it should not be omitted
Omission error from DIR5 model & 2’ gravity/topography grid

- DIR5 $N_{max}$ used: 210 in CA, US; 250 AL; 280 ME

<table>
<thead>
<tr>
<th>Region</th>
<th>Mean (m²s⁻²)</th>
<th>$\sigma$ (m²s⁻²)</th>
<th>Min (m²s⁻²)</th>
<th>Max (m²s⁻²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atlantic Canada</td>
<td>0.75</td>
<td>3.00</td>
<td>−3.53</td>
<td>4.52</td>
</tr>
<tr>
<td>Pacific Canada</td>
<td>−2.21</td>
<td>2.56</td>
<td>−5.82</td>
<td>0.24</td>
</tr>
<tr>
<td>Atlantic USA I</td>
<td>1.05</td>
<td>1.29</td>
<td>−1.38</td>
<td>2.81</td>
</tr>
<tr>
<td>Atlantic USA II</td>
<td>−1.60</td>
<td>2.54</td>
<td>−5.59</td>
<td>1.56</td>
</tr>
<tr>
<td>Atlantic USA III</td>
<td>0.25</td>
<td>2.89</td>
<td>−3.91</td>
<td>4.62</td>
</tr>
<tr>
<td>Pacific USA I</td>
<td>−1.66</td>
<td>2.96</td>
<td>−7.12</td>
<td>3.11</td>
</tr>
<tr>
<td>Pacific USA II</td>
<td>−1.12</td>
<td>1.80</td>
<td>3.49</td>
<td>2.32</td>
</tr>
<tr>
<td>Gulf of Mexico</td>
<td>−0.23</td>
<td>1.59</td>
<td>−3.70</td>
<td>2.86</td>
</tr>
</tbody>
</table>

- Same conclusions as previous test

- Conclusions have also been verified at GNSS / levelling stations in Canada, USA, Mexico
Evaluation of Indirect Effect

- Used mean offsets on a 30’ grid w.r.t. the $W_o = 62,636,853.4\ m^2/s^2$ potential
- Computed at GNSS/levelling stations for various $N_{max}$ values in DIR5 and $S_p^{res}$

Results (on next page) show that:

- $< 1\ cm$ when the satellite-only GGMs are used to $d/o \geq 180$ – can be omitted
- Observation equations are greatly simplified (can set all $f_{Pi} = 0$)
  - Datum offsets are then just the weighted averages of all station $\delta W_{oi}$ values
Indirect bias term computed with the original Stokes kernel > 40 cm!
Indirect Bias Term (2/2)

N_{max}=120

[\text{m}^2 \text{s}^{-2}]

0.20
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
It is insignificant (max value < 1 cm) when the satellite-only GGMs are used to \( d/o \geq 180 \)
**SA Vertical Datum Unification**

*Required data for the formulation of the observation equations*

- Terrestrial gravity anomalies distributed homogeneously and in a high geographical density for the estimation of the disturbing potential $T$.
- Geopotential numbers $C_P$ derived from levelling with gravity corrections in land areas.
  - Optional: geopotential numbers from geostrophic or steric levelling or satellite or altimetry-based mean dynamic topography (MDT) models in ocean areas.
- Ellipsoidal heights $h$ derived from GNSS positioning in land areas and from satellite altimetry in ocean areas.
- Border levelling points with geopotential numbers referring to neighbouring vertical datums $C_{P,i}, C_{P,i+1}$.
Observables available in South America

- 663 geometric reference stations connected with the national vertical datums.
- 7 international connections between neighbouring national vertical datums.
- 14 reference tide gauges with MDT values.
- All these stations with: known geopotential numbers, ellipsoidal heights and anomalous potential values (all of them with uncertainty values).
SA Vertical Datum Unification

**Empirical procedure**

- Harmonisation of the input data, e.g.; all heights in zero tide system and at the same reference epoch, the same reference GGM (n=200) for the estimation of $T$.
- Weighted least-squares adjustment using the inverse of the variances of the input data and rigorous error propagation analysis.

*Standard deviation of the input data used for the vertical datum unification in South America.*
◆ Results

Accuracy:

◆ ±0.5 m²s⁻² (±5 cm) in regions with a high number of observations (Argentina, Brazil, Colombia, Ecuador, Uruguay and Venezuela).

◆ ±2 to ±4 m²s⁻² (±20 to ±40 cm) in regions with a small number of observations (Bolivia, Peru and the southern part of Chile).
Conclusions (1/2)

- The omission error should always be accounted for, in particular in regions with poor distribution of TG or GNSS-levelling stations.

- The indirect bias term can be omitted for a GGM of DO 180 in North America (below 1 cm)
  
  Then the LVD offset is a (weighted) mean of the discrepancies between the geometrically-derived and gravimetric geoid heights.

- As the estimation of the datum parameters should be as reliable as possible, only geodetic stations of highest quality should be considered for the vertical datum unification.

- Possible sources of inconsistency should be removed; i.e., standardised geodetic data is required; for example, geometric coordinates should refer to the same ITRF and be given in the same tide system and reference epoch as the geopotential numbers and gravity field model.
Conclusions (2/2)

- Since the disturbing potential should be estimated with high-precision, it is recommended to compute (a) the long wavelength component \((n \leq 200)\) using a common GGM, and (b) the short wavelength component \((n > 200)\) by the combination of terrestrial gravity data and detailed terrain models. The use of a GGM is not sufficient.

- After a standardisation of the input data used in the unification of the South American height systems and a rigorous error propagation analysis, we demonstrate that the vertical datum parameters can be estimated with accuracy better than \(\pm 5\) cm in well-surveyed regions and some decimetres \((\pm 40\) cm\) in sparsely surveyed regions.

- Once a first estimation of the vertical datum parameters is available, the height-related observables (geopotential numbers, terrestrial gravity anomalies) should be re-computed and used to iterate the GBVP solution. This procedure should be repeated until sub-mm differences are obtained between consecutive iterations.